

# PEIERLS BRACKETS IN FIELD THEORY

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**Abstract.** Peierls brackets are part of the space-time approach to quantum field theory, and provide a Poisson bracket which, being defined for pairs of observables which are group invariant, is group invariant by construction. It is therefore well suited for combining the use of Poisson brackets and the full diffeomorphism group in general relativity. The present paper provides an introduction to the topic, with applications to gauge field theory.

# 1. Introduction

Although the Hamiltonian formalism provides a powerful tool for studying general relativity,<sup>1</sup> its initial-value problem and the approach to canonical quantization,<sup>2</sup> it suffers from severe drawbacks: the space + time split of  $(M, g)$  disagrees with the aims of general relativity, and the space-time topology is taken to be  $\Sigma \times \mathbf{R}$ , so that the full diffeomorphism group of  $M$  is lost.<sup>3,4</sup>

However, as was shown by DeWitt in the sixties,<sup>5</sup> it remains possible to use a Poisson-bracket formalism which preserves the full invariance properties of the original theory, by relying upon the work of Peierls.<sup>6</sup> In our paper, whose aims are pedagogical, we begin by describing the general framework, assuming that the reader has been introduced to the DeWitt covariant approach to quantum field theory.<sup>5</sup> Let us therefore consider a gauge field theory with classical action functional  $S$  and generators of infinitesimal gauge transformations denoted by  $R^i_\alpha$ . The small disturbances  $\delta\varphi^i$  are ruled by the invertible differential operator

$$F_{ij} \equiv S_{,ij} + \gamma_{ik} R^k_\alpha \tilde{\gamma}^{\alpha\beta} \gamma_{jl} R^l_\beta, \quad (1)$$

where  $\gamma_{ij}$  is a local and symmetric matrix which is taken to transform like  $S_{,ij}$  under group transformations, and  $\tilde{\gamma}^{\alpha\beta}$  is a local, non-singular, symmetric matrix which transforms according to the adjoint representation of the infinite-dimensional invariance group (hence one gets  $R_{i\alpha} \equiv \gamma_{ij} R^j_\alpha$  and  $R_i^\alpha \equiv \tilde{\gamma}^{\alpha\beta} R_{i\beta}$ , respectively). We are interested in advanced and retarded Green functions  $G^\pm$  which are left inverses of  $-F$ , i.e.

$$G^{\pm ij} F_{jk} = -\delta^i_k. \quad (2)$$

Furthermore, the form of  $F_{ij}$  and arbitrariness of Cauchy data imply that  $G^\pm$  are right inverses as well, i.e.

$$F_{ij} G^{\pm jk} = -\delta_i^k. \quad (3)$$

If symmetry of  $F$  is required, one also finds

$$G^{+ij} = G^{-ji}, \quad G^{-ij} = G^{+ji}, \quad (4)$$

because in general

$$G^{\pm ij} - G^{\mp ji} = G^{\pm ik}(F_{kl} - F_{lk})G^{\mp jl}. \quad (5)$$

Thus, the *supercommutator function* defined as

$$\tilde{G}^{ij} \equiv G^{+ij} - G^{-ij} \quad (6)$$

is antisymmetric in that  $\tilde{G}^{ij} = -\tilde{G}^{ji}$ . These properties show that, on defining  $\delta_A^\pm B \equiv \varepsilon B_{,i} G^{\pm ij} A_{,j}$ , one has, on relabelling dummy indices,

$$\delta_A^\pm B = \varepsilon B_{,j} G^{\pm ji} A_{,i} = \varepsilon A_{,i} G^{\mp ij} B_{,j} = \delta_B^\mp A. \quad (7)$$

These are the *reciprocity relations*, which express the idea that the retarded (resp. advanced) effect of  $A$  on  $B$  equals the advanced (resp. retarded) effect of  $B$  on  $A$ . Another cornerstone of the formalism is a relation involving the Green function  $\hat{G}$  of the operator  $-\hat{F}$ , having set  $R_{k\beta} R_\alpha^k \equiv \hat{F}_{\beta\alpha}$ ; this is

$$R_\alpha^i \hat{G}^{\pm\alpha\beta} \tilde{\gamma}_{\beta\delta} = R_\alpha^i \hat{G}^{\pm\alpha}_\delta = G^{\pm ij} \gamma_{jk} R_\delta^k = G^{\pm ij} R_{j\delta}. \quad (8)$$

This holds because, *for background fields satisfying the field equations*, one finds that

$$F_{ik} R_\alpha^k = R_i^\beta R_{k\beta} R_\alpha^k = R_i^\beta \hat{F}_{\beta\alpha}. \quad (9)$$

On multiplying this equation on the left by  $G^{\pm ji}$  and on the right by  $\hat{G}^{\pm\alpha\beta}$  one gets

$$R_\alpha^j \hat{G}^{\pm\alpha\beta} = G^{\pm ji} R_i^\beta, \quad (10)$$

i.e. the desired formula (8) is proved. Moreover, by virtue of (4), the transposed equations

$$\hat{G}^{\pm\alpha\beta} R_\beta^j = R_i^\alpha G^{\pm ij} \quad (11)$$

also hold. We are now in a position to define the Peierls bracket of any two observables  $A$  and  $B$ . First, we consider the operation

$$D_A B \equiv \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \delta_A^- B, \quad (12)$$

with  $D_B A$  obtained by interchanging  $A$  with  $B$  in (12). The *Peierls bracket* of  $A$  and  $B$  is then defined by

$$(A, B) \equiv D_A B - D_B A = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left[ \varepsilon A_1 G^+ B_1 - \varepsilon A_1 G^- B_1 \right] = A_1 \tilde{G} B_1 = A_{,i} \tilde{G}^{ij} B_{,j}, \quad (13)$$

where we have used (7) and (12) to obtain the last expression. Following DeWitt,<sup>7</sup> it should be stressed that the Peierls bracket depends only on the behaviour of infinitesimal disturbances.

In classical mechanics, following Peierls,<sup>6</sup> we may arrive at the derivatives in (12) and (13) starting from the action functional  $S \equiv \int L d\tau$  and considering the extremals of  $S$  and those of  $S + \lambda A$ , where  $\lambda$  is an infinitesimal parameter and  $A$  any function of the path  $\gamma$ . Next we consider solutions of the modified equations as expansions in powers of  $\lambda$ , and hence the new set of solutions to first order reads

$$\gamma'(\tau) = \gamma(\tau) + \lambda D_A \gamma(\tau). \quad (14)$$

This modified solution is required to obey the condition that, in the distant past, it should be identical with the original one, i.e.

$$D_A \gamma(\tau) \rightarrow 0 \text{ as } \tau \rightarrow -\infty. \quad (15)$$

Similarly to the construction of the above “retarded” solution, we may define an “advanced” solution

$$\gamma''(\tau) = \gamma(\tau) + \lambda \mathcal{D}_A \gamma(\tau), \quad (16)$$

such that

$$\mathcal{D}_A \gamma(\tau) \rightarrow 0 \text{ as } \tau \rightarrow +\infty. \quad (17)$$

From these modified solutions one can now find  $D_A \gamma(\tau)$  along the solutions of the unmodified action and therefore, to first order, the changes in any other function  $B$  of the field variables, and these are denoted by  $D_A B$  and  $D_B A$ , respectively.

## 2. Mathematical Properties of Peierls Brackets

We are now aiming to prove that  $(A, B)$  satisfies all properties of a Poisson bracket. The first two, anti-symmetry and bilinearity, are indeed obvious:

$$(A, B) = -(B, A), \quad (18)$$

$$(A, B + C) = (A, B) + (A, C), \quad (19)$$

whereas the proof of the Jacobi identity is not obvious and is therefore presented in detail.

First, by repeated application of (13) one finds

$$\begin{aligned} P(A, B, C) &\equiv (A, (B, C)) + (B, (C, A)) + (C, (A, B)) \\ &= A_{,i} \tilde{G}^{il} \left( B_{,j} \tilde{G}^{jk} C_{,k} \right)_{,l} + B_{,j} \tilde{G}^{jl} \left( C_{,k} \tilde{G}^{ki} A_{,i} \right)_{,l} + C_{,k} \tilde{G}^{kl} \left( A_{,i} \tilde{G}^{ij} B_{,j} \right)_{,l} \\ &= A_{,il} B_{,j} C_{,k} \left( \tilde{G}^{ij} \tilde{G}^{kl} + \tilde{G}^{jl} \tilde{G}^{ki} \right) + A_{,i} B_{,jl} C_{,k} \left( \tilde{G}^{jk} \tilde{G}^{il} + \tilde{G}^{kl} \tilde{G}^{ij} \right) \\ &\quad + A_{,i} B_{,j} C_{,kl} \left( \tilde{G}^{ki} \tilde{G}^{jl} + \tilde{G}^{il} \tilde{G}^{jk} \right) \\ &\quad + A_{,i} B_{,j} C_{,k} \left( \tilde{G}^{il} \tilde{G}^{jk} \right)_{,l} + \tilde{G}^{jl} \tilde{G}^{ki} \left( A_{,i} \right)_{,l} + \tilde{G}^{kl} \tilde{G}^{ij} \left( B_{,j} \right)_{,l}. \end{aligned} \quad (20)$$

Now the antisymmetry property of  $\tilde{G}$ , jointly with commutation of functional derivatives:  $T_{,il} = T_{,li}$  for all  $T = A, B, C$ , implies that the first three terms on the last equality in (20) vanish. For example one finds

$$\begin{aligned} A_{,il} B_{,j} C_{,k} \left( \tilde{G}^{ij} \tilde{G}^{kl} + \tilde{G}^{jl} \tilde{G}^{ki} \right) &= A_{,li} B_{,j} C_{,k} \left( \tilde{G}^{lj} \tilde{G}^{ki} + \tilde{G}^{ji} \tilde{G}^{kl} \right) \\ &= -A_{,il} B_{,j} C_{,k} \left( \tilde{G}^{jl} \tilde{G}^{ki} + \tilde{G}^{ij} \tilde{G}^{kl} \right) = 0, \end{aligned} \quad (21)$$

and an entirely analogous procedure can be applied to the terms containing the second functional derivatives  $B_{,jl}$  and  $C_{,kl}$ . The last term in (20) requires new calculations because it contains functional derivatives of  $\tilde{G}^{ij}$ . These can be dealt with after taking infinitesimal variations of Eq. (3), so that

$$F \delta G^{\pm} = -(\delta F) G^{\pm}, \quad (22)$$

and hence

$$G^\pm F \delta G^\pm = F G^\pm \delta G^\pm = -\delta G^\pm = -G^\pm (\delta F) G^\pm, \quad (23)$$

i.e.

$$\delta G^\pm = G^\pm (\delta F) G^\pm. \quad (24)$$

Thus, the desired functional derivatives of advanced and retarded Green functions read

$$\begin{aligned} G^{\pm ij}_{,c} &= G^{\pm ia} F_{ab,c} G^{\pm bj} = G^{\pm ia} \left( S_{,ab} + R_{a\alpha} R_b^\alpha \right)_{,c} G^{\pm bj} \\ &= G^{\pm ia} S_{,abc} G^{\pm bj} + G^{\pm ia} R_{a\alpha,c} R_b^\alpha G^{\pm bj} + G^{\pm ia} R_{a\alpha} R_b^\alpha_{,c} G^{\pm bj}. \end{aligned} \quad (25)$$

In this formula the contractions  $R_b^\alpha G^{\pm bj}$  and  $G^{\pm ia} R_{a\alpha}$  can be re-expressed with the help of Eqs. (10) and (11), and eventually one gets

$$G^{\pm ij}_{,c} = G^{\pm ia} S_{,abc} G^{\pm bj} + G^{\pm ia} R_{a\alpha,c} \hat{G}^{\pm\alpha\beta} R_\beta^j + R_\beta^i \hat{G}^{\pm\beta}_\alpha R_b^\alpha_{,c} G^{\pm bj}. \quad (26)$$

By virtue of the group invariance property satisfied by all physical observables, the second and third term on the right-hand side of Eq. (26) give vanishing contribution to (20). One is therefore left with the contributions involving third functional derivatives of the action. Bearing in mind that  $S_{,abc} = S_{,acb} = S_{,bca} = \dots$ , one can relabel indices summed over, finding eventually (upon using (4))

$$\begin{aligned} P(A, B, C) &= A_{,i} B_{,j} C_{,k} \left[ (G^{+ic} - G^{-ic})(G^{+ja} G^{+bk} - G^{-ja} G^{-bk}) \right. \\ &\quad + (G^{+jc} - G^{-jc})(G^{+ka} G^{+bi} - G^{-ka} G^{-bi}) \\ &\quad \left. + (G^{+kc} - G^{-kc})(G^{+ia} G^{+bj} - G^{-ia} G^{-bj}) \right] S_{,abc} \\ &= A_{,i} B_{,j} C_{,k} \left[ (G^{+ia} - G^{-ia})(G^{+jb} G^{-kc} - G^{-jb} G^{+kc}) \right. \\ &\quad + (G^{+jb} - G^{-jb})(G^{+kc} G^{-ia} - G^{-kc} G^{+ia}) \\ &\quad \left. + (G^{+kc} - G^{-kc})(G^{+ia} G^{-jb} - G^{-ia} G^{+jb}) \right] S_{,abc} = 0. \end{aligned} \quad (27)$$

This sum vanishes because it involves six pairs of triple products of Green functions with opposite signs. The Jacobi identity is therefore fulfilled. Moreover, the fourth fundamental property of Poisson brackets, i.e.

$$(A, BC) = (A, B)C + B(A, C) \quad (28)$$

is also satisfied, because

$$(A, BC) = A_{,i} \tilde{G}^{ik} (BC)_{,k} = A_{,i} \tilde{G}^{ik} B_{,k} C + B A_{,i} \tilde{G}^{ik} C_{,k} = (A, B)C + B(A, C). \quad (29)$$

Thus, the Peierls bracket defined in (13) is indeed a Poisson bracket of physical observables. Equation (28) can be regarded as a compatibility condition of the Peierls bracket with the product of physical observables.

It should be stressed that the idea of Peierls<sup>6</sup> was to introduce a bracket related directly to the action principle without making any reference to the Hamiltonian. This implies that even classical mechanics should be considered as a “field theory” in a zero-dimensional space, having only the time dimension. This means that one deals with an infinite-dimensional space of paths  $\gamma : \mathbf{R} \rightarrow Q$ , therefore we are dealing with functional derivatives and distributions even in this situation where modern standard treatments rely upon  $C^\infty$  manifolds and smooth structures. Thus, the present treatment is hiding most technicalities involving infinite-dimensional manifolds. In finite dimensions on a smooth manifold, any bracket satisfying (19) and (28) is associated with first-order bidifferential operators;<sup>8,9</sup> in this proof it is important that the commutative and associative product  $BC$  is a local product. In any case these brackets at the classical level could be a starting point to define a  $*$ -product in the spirit of non-commutative geometry<sup>10</sup> or deformation quantization.<sup>11</sup>

### 3. The most general Peierls bracket

The Peierls bracket is a group invariant by construction, being defined for pairs of observables which are group invariant, and is invariant under both infinitesimal and finite

changes in the matrices  $\gamma_{ij}$  and  $\tilde{\gamma}_{\alpha\beta}$ . DeWitt [5] went on to prove that, even if independent differential operators  $P_i^\alpha$  and  $Q_{i\alpha}$  are introduced such that

$$F_{ij} \equiv S_{,ij} + P_i^\alpha Q_{j\alpha}, \quad \hat{F}_{\alpha\beta} \equiv Q_{i\alpha} R_\beta^i, \quad F_\alpha^\beta \equiv R_\alpha^i P_i^\beta, \quad (30)$$

are all non-singular, with unique advanced and retarded Green functions, the reciprocity theorem expressed by (7) still holds, and the resulting Peierls bracket is invariant under changes in the  $P_i^\alpha$  and  $Q_{i\alpha}$ , by virtue of the identities

$$Q_{i\alpha} G^{\pm ij} = G_\alpha^{\pm\beta} R_\beta^j, \quad (31)$$

$$G^{\pm ij} P_j^\beta = R_\alpha^i \hat{G}^{\pm\alpha\beta}. \quad (32)$$

This is proved as follows. The composition of  $F_{ik}$  with the infinitesimal generators of gauge transformations yields

$$F_{ik} R_\alpha^k = P_i^\beta F_{\beta\alpha}, \quad (33)$$

and hence

$$G^{\pm ji} F_{ik} R_\alpha^k = -R_\alpha^j = G^{\pm ji} P_i^\gamma F_{\gamma\alpha}, \quad (34)$$

which implies

$$R_\alpha^j G^{\pm\alpha\beta} = -G^{\pm ji} P_i^\gamma F_{\gamma\alpha} G^{\pm\alpha\beta} = G^{\pm ji} P_i^\beta, \quad (35)$$

i.e. Eq. (32) is obtained. Similarly,

$$R_\alpha^i F_{ij} = F_\alpha^\beta Q_{j\beta}, \quad (36)$$

and hence

$$G_\alpha^{\pm\gamma} R_\gamma^i F_{ij} = -Q_{j\alpha}, \quad (37)$$

which implies

$$Q_{i\alpha} G^{\pm ij} = -G_\alpha^{\pm\gamma} R_\gamma^k F_{ki} G^{\pm ij} = G_\alpha^{\pm\beta} R_\beta^j, \quad (38)$$

i.e. Eq. (31) is obtained. Now we use the first line of Eq. (7) for  $\delta_A^\pm B$ , jointly with Eq. (5), so that

$$\delta_A^\pm B - \varepsilon B_{,i} G^{\mp ji} A_{,j} = \varepsilon B_{,i} R_\gamma^i G^{\pm\gamma\alpha} Q_{l\alpha} G^{\mp jl} A_{,j} - \varepsilon B_{,i} P_l^\alpha G^{\pm ik} Q_{k\alpha} G^{\mp jl} A_{,j}. \quad (39)$$



Since  $B$  is an observable by hypothesis, the first term on the right-hand side of (39) vanishes. Moreover one finds, from (32)

$$G^{\pm ik} P_l^\alpha Q_{k\alpha} G^{\mp jl} = G^{\pm il} R_\beta^j G^{\mp \beta\alpha} Q_{l\alpha}. \quad (40)$$

and hence also the second term on the right-hand side of (39) vanishes ( $A$  being an observable, for which  $R_\beta^j A_{,j} = 0$ ), yielding eventually the reciprocity relation (7). Moreover, the invariance of the Peierls bracket under variations of  $P_{i\alpha}$  and  $Q_i^\alpha$  holds because

$$\begin{aligned} \delta(\delta_A^\pm B) &= \varepsilon B_{,i} \delta G^{\pm ij} A_{,j} = \varepsilon B_{,i} G^{\pm ik} (\delta F_{kl}) G^{\pm lj} A_{,j} \\ &= \varepsilon B_{,i} G^{\pm ik} \left[ (\delta P_k^\alpha) Q_{l\alpha} + P_k^\alpha (\delta Q_{l\alpha}) \right] G^{\pm lj} A_{,j} \\ &= \varepsilon B_{,i} G^{\pm ik} (\delta P_k^\alpha) Q_{l\alpha} G^{\pm lj} A_{,j} + \varepsilon B_{,i} G^{\pm ik} P_k^\alpha (\delta Q_{l\alpha}) G^{\pm lj} A_{,j} \\ &= \varepsilon B_{,i} G^{\pm ik} (\delta P_k^\alpha) G_\alpha^\pm R_\beta^j A_{,j} + \varepsilon B_{,i} R_\gamma^i G^{\pm \gamma\alpha} (\delta Q_{l\alpha}) G^{\pm lj} A_{,j} = 0, \end{aligned} \quad (41)$$

where Eqs. (31) and (32) have been exploited once more.

#### 4. Concluding Remarks

The Peierls-bracket formalism is equivalent to the conventional canonical formalism when the latter exists. The proof can be given starting from point Lagrangians, as is shown in Ref. 5. Current applications of Peierls brackets deal with string theory,<sup>12,13</sup> path integration and decoherence,<sup>14</sup> supersymmetric proof of the index theorem,<sup>15</sup> classical dynamical systems involving parafermionic and parabosonic dynamical variables,<sup>16</sup> while for recent literature on covariant approaches to a canonical formulation of field theories we refer the reader to the work in Refs. 17–24.

In the infinite-dimensional setting which, strictly, applies also to classical mechanics, as we stressed at the end of Sec. 2, we hope to elucidate the relation between a covariant description of dynamics as obtained from the kernel of the symplectic form, and a parametrized description of dynamics as obtained from any Poisson bracket, including the Peierls bracket.

## Acknowledgements

The work of the authors has been partially supported by PRIN 2000 and PRIN 2002 *Sintesi*.

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